Local approximation of the solutions of algebraic equations

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Abstract. We give a constructive proof of the fact that every holomorphic solution $y = f(x)$ of a system $Q(x, y) = 0$ of polynomial (or Nash) equations can be approximated in a fixed neighborhood of every $x_0 \in dom(f)$ by a sequence of Nash solutions. An algorithm of such approximation is next presented.

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1 Introduction

The aim of this note is to present a simple geometric proof of the following approximation theorem. The proof allows to construct an algorithm which can be used in numerical computations (see Section 3.2).

Theorem 1.1 *Let U be an open subset of $\mathbf{C}^n$ and let $f : U \to \mathbf{C}^k$ be a holomorphic mapping that satisfies a system of equations $Q(x, f(x)) = 0$ for $x \in U$, where $Q : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^q$ is a polynomial mapping. Then for every $x_0 \in U$ there are an open neighborhood $U_0 \subset U$ and a sequence $\{f^\nu : U_0 \to \mathbf{C}^k\}$ of Nash mappings converging uniformly to $f|_{U_0}$ such that $Q(x, f^\nu(x)) = 0$ for every $x \in U_0$ and $\nu \in \mathbf{N}$.*

Local approximation of the solutions of algebraic or analytic equations was investigated by M. Artin in [2], [3] and [4] and Theorem 1.1 can be derived from the results of these papers. Since our goal is to obtain an effective procedure of approximation, the present note treats the problem in a bit different way: first a reduction to the case where Q is a single polynomial (which, given the original equations, can be computed) is carried out. The reduced problem is next solved and here the combination of some of the ideas of [2] and [11] is applied (see Section 3.1). The latter article, due to L. van den Dries, deals with the global version of the theorem for f depending on one variable.

Theorem 1.1 can be easily strengthened by replacing the polynomial mapping Q by a Nash mapping defined in some neighborhood of the graph of f (cf. Section 3.1). The method of approximation presented in this note is more efficient than the one we gave in [8], where the question of the existence of mappings approximating solutions of algebraic equations is also discussed.

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Both in the real case (M. Coste, J. M. Ruiz, M. Shiota [10]) and in the complex case (L. Lempert [12]) more general (global) versions of Theorem 1.1 are known to be true. These results rely on the solution to the deep and important M. Artin's conjecture for which the reader is referred to [1], [13], [14], [15], [16]. Such an approach enabled the authors to reach the goal in an elegant and relatively short way, but makes the proofs difficult to be applied in effective computations.

Our interest in Theorem 1.1 is partially motivated by applications in the theory of analytic sets. In particular, papers [5]–[7] contain results on approximation of complex analytic sets by complex Nash sets whose proofs can be divided into two stages: (i) preparation, where only direct geometric methods appear, (ii) switching Theorem 1.1. Thus the techniques of the present article allow to obtain local versions of these facts in a purely geometric way.

The paper is organized as follows. Sections 3.1 and 3.2 are devoted to Theorem 1.1 and the algorithm respectively. Preliminary material concerning Nash mappings and sets as well as analytic sets with proper projection is gathered in Section 2 below.

2 Preliminaries

2.1 Nash mappings and sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \ldots, f_s$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \ldots = f_s(x) = 0\}.$$

We will use the following fact from [17], p. 239. Let $\pi : \Omega \times \mathbf{C}^k \to \Omega$ denote the natural projection.

Theorem 2.1 *Let X be a Nash subset of $\Omega \times \mathbf{C}^k$ such that $\pi|_X : X \to \Omega$ is a proper mapping. Then $\pi(X)$ is a Nash subset of Ω and $dim(X) = dim(\pi(X))$.*

The fact from [17] stated below explains the relation between Nash and algebraic sets.

Theorem 2.2 *Let X be a Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then every analytic irreducible component of X is an irreducible Nash subset of Ω. Moreover, if X is irreducible then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$.*

2.2 Analytic sets

Let U, U' be domains in $\mathbf{C}^n, \mathbf{C}^k$ respectively and let $\pi : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^n$ denote the natural projection. For any purely n-dimensional analytic subset Y of $U \times U'$ such that $\pi|_Y : Y \to U$ is a proper mapping by $\mathcal{S}(Y, \pi)$ we denote the set of singular points of $\pi|_Y$:

$$\mathcal{S}(Y, \pi) = Sing(Y) \cup \{x \in Reg(Y) : (\pi|_Y)'(x) \text{ is not an isomorphism}\}.$$

We often write $\mathcal{S}(Y)$ instead of $\mathcal{S}(Y, \pi)$ when it is clear which projection is taken into consideration.

It is well known that $\mathcal{S}(Y)$ is an analytic subset of $U \times U'$, $dim(Y) < n$ (cp. [9], p. 50), hence by the Remmert theorem $\pi(\mathcal{S}(Y))$ is also analytic. Moreover, the following hold. The mapping $\pi|_Y$ is surjective and open and there exists an integer $s = s(\pi|_Y)$ such that:

(1) $\sharp(\pi|_Y)^{-1}(\{a\}) < s$ for $a \in \pi(\mathcal{S}(Y))$,
(2) $\sharp(\pi|_Y)^{-1}(\{a\}) = s$ for $a \in U \setminus \pi(\mathcal{S}(Y))$,
(3) for every $a \in U \setminus \pi(\mathcal{S}(Y))$ there exists a neighborhood W of a and holomorphic mappings $f_1, \ldots, f_s : W \to U'$ such that $f_i \cap f_j = \emptyset$ for $i \neq j$ and $f_1 \cup \ldots \cup f_s = (W \times U') \cap Y$.

Let E be a purely n-dimensional analytic subset of $U \times U'$ with proper projection onto a domain $U \subset \mathbf{C}^n$, where U' is a domain in $\mathbf{C}$. Then there is a unitary polynomial $p \in \mathcal{O}(U)[z]$ such that $E = \{(x, z) \in U \times \mathbf{C} : p(x, z) = 0\}$ and the discriminant Δ_p of p is not identically zero. p will be called the optimal polynomial for E. It holds: $\tilde{\pi}(\mathcal{S}(E)) = \{x \in U : \Delta_p(x) = 0\}$, where $\tilde{\pi} : U \times \mathbf{C} \to U$ is the natural projection.

Finally, for any analytic subset X of an open set $\tilde{U} \subset \mathbf{C}^m$ let $X_{(k)} \subset \tilde{U}$ denote the union of all irreducible components of X of dimension k.

3 Approximation

3.1 Proof of Theorem 1.1

Theorem 1.1, formulated in Section 1, is equivalent to its slightly stronger version in which Q is a Nash mapping defined on some neighborhood of the graph of f. Both versions are in turn equivalent to the following

Proposition 3.1 *Let U, V be a domain in $\mathbf{C}^n$ and an algebraic subvariety of $\mathbf{C}^{\hat{m}}$ respectively. Let $F : U \to V$ be a holomorphic mapping. Then for every $x_0 \in U$ there are an open neighborhood $U_0 \subset U$ and a sequence $\{F^\nu : U_0 \to V\}$ of Nash mappings converging uniformly to $F|_{U_0}$.*

The fact that Proposition 3.1 does imply (the stronger version of) Theorem 1.1 is well known (see [12]) and we prove it below only for completeness. The converse implication is clear.

Let $f : U \to \mathbf{C}^k$ be the holomorphic mapping from Theorem 1.1. Assume that Q is a Nash mapping defined on some neighborhood $\hat{U}$ of the graph of f

in $\mathbf{C}^n \times \mathbf{C}^k$, such that $Q(x, f(x)) = 0$ for $x \in U$. Next put $F(x) = (x, f(x))$ and $\hat{m} = n + k$. Let V be the intersection of all algebraic subvarieties of $\mathbf{C}^{\hat{m}}$ containing $F(U)$. Then by Proposition 3.1 there is a sequence $\{F^\nu : U_1 \to V\}$ of Nash mappings converging to $F|_{U_1}$, where $U_1 \subset U$ is a neighborhood of a fixed x_0.

We need to show that the first n components of F^ν may be assumed to constitute the identity and that $Q \circ F^\nu = 0$ for sufficiently large ν. To this end denote $Y = \{(x, v) \in \hat{U} \subset \mathbf{C}^n \times \mathbf{C}^k : Q(x, v) = 0\}$. Clearly, we may assume that $F^\nu(U_1) \subset \hat{U}$ for almost all ν. Next observe that $F^\nu(U_1) \subset Y$ for almost all ν. Indeed, take $\hat{z} \in F(U_1) \cap Reg(V)$ (the intersection is non-empty as $F(U_1) \subset Sing(V)$ implies, by the connectedness of U, that $F(U) \subset Sing(V) \subsetneq V$). Let B be an open neighborhood of $\hat{z}$ in $\mathbf{C}^n \times \mathbf{C}^k$ such that $B \cap V$ is a connected manifold and let U_2 be a non-empty open subset of U_1 such that $F(U_2), F^\nu(U_2) \subset B$ for almost all ν. Then $B \cap V \subset Y$ (otherwise $F(U_2) \subset \tilde{V}$ where $\tilde{V}$ is an algebraic subvariety of $\mathbf{C}^n \times \mathbf{C}^k$ with $dim(\tilde{V}) < dim(V)$). This implies that $F^\nu(U_2) \subset Y$ for almost all ν hence $F^\nu(U_1) \subset Y$ because U_1 is connected.

Let $\tilde{F}^\nu : U_1 \to \mathbf{C}^n$, for $\nu \in \mathbf{N}$, be the mapping whose components are the first n components of F^ν. Take a neighborhood $U_0 \subset\subset U_1$ of x_0. Since $\{\tilde{F}^\nu\}$ converges uniformly to the identity on U_1 and $U_0 \subset\subset U_1$ there is a sequence $H^\nu : U_0 \to U_1$ of Nash mappings such that $\tilde{F}^\nu \circ H^\nu = id_{U_0}$ if ν is large enough. Consequently, $F^\nu \circ H^\nu(x) = (x, f^\nu(x))$ for $x \in U_0$ and $\{f^\nu : U_0 \to \mathbf{C}^k\}$ satisfies the assertion of Theorem 1.1.

Proof of Proposition 3.1. First observe that since U is connected, $F(U)$ is contained in one irreducible component of V so we may assume that V is of pure dimension, say m.

We may also assume that $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ is with proper projection onto $\mathbf{C}^m$. Indeed, for the generic $\mathbf{C}$-linear isomorphism $J : \mathbf{C}^{m+s} \to \mathbf{C}^{m+s}$ the image $J(V)$ is with proper projection onto $\mathbf{C}^m$. Thus if there exists a sequence $H^\nu : U_0 \to J(V)$ of Nash mappings converging to $J \circ F|_{U_0}$ then the sequence $\{J^{-1} \circ H^\nu\}$ satisfies the assertion of the proposition.

Now the problem is reduced to the case where V is a hypersurface (compare [11], p. 394). Any $\mathbf{C}$-linear form $L : \mathbf{C}^s \to \mathbf{C}$ determines the mapping $\Phi_L : \mathbf{C}^m \times \mathbf{C}^s \to \mathbf{C}^m \times \mathbf{C}$ by the formula $\Phi_L(u, v) = (u, L(v))$. Since V is an algebraic subset of $\mathbf{C}^m \times \mathbf{C}^s$ with proper projection onto $\mathbf{C}^m$ then $\Phi_L(V)$ is an algebraic subset of $\mathbf{C}^m \times \mathbf{C}$ also with proper projection onto $\mathbf{C}^m$ for every form L. Take L such that the cardinalities of the fibers of the projections of $\Phi_L(V)$ and V onto $\mathbf{C}^m$ are equal over almost every point of $\mathbf{C}^m$. (The generic L has this property.) The set $\Phi_L(V)$ is described by the unitary polynomial P_L in one variable (corresponding to the last coordinate of $\mathbf{C}^m \times \mathbf{C}$) whose coefficients are polynomials in m variables and whose discriminant is non-zero (see Section 2.2).

Let $f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+s}$ denote the coordinates of the mapping F. We may assume that $R_L(f_1, \ldots, f_m) \neq 0$, where R_L denotes the discriminant of P_L. Indeed, otherwise we return to the very beginning of the proof with V replaced by $V \cap \{R_L = 0\}$ which also contains the image of F. Since the latter variety is of pure dimension $m - 1$, the procedure must stop.

Put $\tilde{f} = L(f_{m+1}, \dots, f_{m+s})$. Now, F, L, P_L, V satisfy the following lemma which will be useful to us.

Lemma 3.2 *Let $\{f_1^\nu\}, \dots, \{f_m^\nu\}, \{\tilde{f}^\nu\}$ be sequences of holomorphic functions converging locally uniformly to $f_1, \dots, f_m, \tilde{f}$ respectively such that*

$$P_L(f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu) = 0, \text{ for every } \nu \in \mathbf{N}.$$

Then there exist sequences of holomorphic functions $\{f_{m+1}^\nu\}, \dots, \{f_{m+s}^\nu\}$ converging locally uniformly to $f_{m+1}, \dots, f_{m+s}$ respectively such that the image of the mapping $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ is contained in V.

Proof of Lemma 3.2. For any holomorphic mapping $H : E \to \mathbf{C}^m$, where E is an open subset of $\mathbf{C}^n$ and any algebraic subvariety X of $\mathbf{C}^m \times \mathbf{C}^s$ denote

$$\mathcal{V}(X, H) = \{(x, v) \in E \times \mathbf{C}^s : (H(x), v) \in X\}.$$

Next put $\Psi_L(x, v) = (x, L(v))$ for any $x \in \mathbf{C}^n, v \in \mathbf{C}^s$. Let $\tilde{\pi} : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^n$, $\pi : \mathbf{C}^n \times \mathbf{C}^s \to \mathbf{C}^n$ denote the natural projections. Assume the notation of Section 2.2. Then the following remark is clearly true.

Remark 3.3 Let $Z \subset E \times \mathbf{C}^s$ be an analytic subset of pure dimension n with proper projection onto a domain $E \subset \mathbf{C}^n$ such that $s(\pi|_Z) = s(\tilde{\pi}|_{\Psi_L(Z)})$. Then for every irreducible analytic component Σ of $\Psi_L(Z)$ there exists an irreducible analytic component Γ of Z such that $\Psi_L(\Gamma) = \Sigma$ and $s(\pi|_\Gamma) = s(\tilde{\pi}|_\Sigma)$.

The remark allows to complete the proof of Lemma 3.2. Put $\tilde{F} = (f_1, \dots, f_m)$, $\tilde{F}^\nu = (f_1^\nu, \dots, f_m^\nu)$, $G = (f_{m+1}, \dots, f_{m+s})$. First observe that the fact that $R_L \circ \tilde{F} \neq 0$ and the way L has been chosen imply that the cardinalities of the generic fibers in $\Psi_L(\mathcal{V}(V, \tilde{F}))$, $\mathcal{V}(V, \tilde{F})$, $\Psi_L(\mathcal{V}(V, \tilde{F}^\nu))$ and in $\mathcal{V}(V, \tilde{F}^\nu)$ over U are equal for large ν. Therefore we may apply Remark 3.3 with $Z = \mathcal{V}(V, \tilde{F}^\nu)$ (for large ν) to obtain the mapping $G^\nu : U \to \mathbf{C}^s$ such that $graph(G^\nu) \subset \mathcal{V}(V, \tilde{F}^\nu)$ and $L \circ G^\nu = \tilde{f}^\nu$.

Observe that $\{G^\nu\}$ converges to G locally uniformly. Indeed, since $\{G^\nu\}$ is locally uniformly bounded then taking any compact subset K of U and passing to a subsequence we may assume that $\{G^\nu\}$ has a limit $\bar{G}$ on K with $graph(\bar{G}) \subset \mathcal{V}(V, \tilde{F})$. Assumption that $\bar{G} \neq G|_K$, the fact that $\Psi_L(\mathcal{V}(V, \tilde{F})) \cap (\{x\} \times \mathbf{C})$ and $\mathcal{V}(V, \tilde{F}) \cap (\{x\} \times \mathbf{C}^s)$ have the same number of elements for almost every $x \in U$ and the facts that $graph(\bar{G}) \subset \mathcal{V}(V, \tilde{F})$ and $L \circ \bar{G} = L \circ G$ give a contradiction.$\blacksquare$

Proof of Proposition 3.1 (continuation). Without loss of generality assume that $x_0 = 0 \in \mathbf{C}^n$. To complete the proof of Proposition 3.1 it is sufficient to show that there are sequences of Nash functions $\{f_1^\nu\}, \dots, \{f_m^\nu\}, \{\tilde{f}^\nu\}$ converging to $f_1, \dots, f_m, \tilde{f}$ respectively, in some neighborhood of 0, such that $P_L(f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu) = 0$. Then by Lemma 3.2 there are sequences of holomorphic functions $\{f_{m+1}^\nu\}, \dots, \{f_{m+s}^\nu\}$ converging to $f_{m+1}, \dots, f_{m+s}$ respectively with the image of $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ contained in V. Therefore

$$graph(f_{m+1}^\nu, \dots, f_{m+s}^\nu) \subset \mathcal{V}(V, (f_1^\nu, \dots, f_m^\nu))$$

so for every $i = 1, \ldots, s$ it holds

$$graph(f_{m+i}^{\nu}) \subset \pi_i(\mathcal{V}(V, (f_1^{\nu}, \ldots, f_m^{\nu})))$$

where $\pi_i : \mathbf{C}^m \times \mathbf{C}_1 \times \ldots \times \mathbf{C}_s \to \mathbf{C}^m \times \mathbf{C}_i$ denotes the natural projection. Consequently, f_{m+i}^{ν} is a Nash function as $\pi_i(\mathcal{V}(V, (f_1^{\nu}, \ldots, f_m^{\nu})))$ is a Nash hypersurface (see Section 2.1).

Let us show that there are sequences of Nash functions $\{f_1^{\nu}\}, \ldots, \{f_m^{\nu}\}, \{\tilde{f}^{\nu}\}$ converging to $f_1, \ldots, f_m, \tilde{f}$ respectively such that $P_L(f_1^{\nu}, \ldots, f_m^{\nu}, \tilde{f}^{\nu}) = 0$ (i.e. let us prove Proposition 3.1 in the simplified situation where V is a hypersurface and the image of F is not contained in $Sing(V)$). This will be done by induction on n (the number of the variables F depends on). Below the reduction to a lower dimensional case (by applying the Weierstrass preparation theorem) will be carried out in a similar way as in [2]. The Tougeron implicit functions theorem, which often appears in the context, is replaced here, as in [11], by the following lemma.

Let $B_n(r)$ denote a compact ball in $\mathbf{C}^n$ of radius r.

Lemma 3.4 ([11], p. 393). *Let d be a positive integer and let M, r be positive real numbers. There is $\varepsilon > 0$ such that for all $A = a_0 z^d + a_1 z^{d-1} + \ldots + a_d \in \mathcal{O}(B_n(r))[z]$ with $\sup_{x \in B_n(r)} |a_i(x)| < M$ where $i = 0, \ldots, d$ and for all $\alpha, c \in \mathcal{O}(B_n(r))$ with $\sup_{x \in B_n(r)} |\alpha(x)| < M$, $\sup_{x \in B_n(r)} |c(x)| < \varepsilon$ such that $A \circ \alpha = c \cdot (\frac{\partial A}{\partial z} \circ \alpha)^2$ the following holds: there is $b \in \mathcal{O}(B_n(r))$ with $A \circ b = 0$ and $\sup_{x \in B_n(r)} |b(x) - \alpha(x)| \leq 2 \sup_{x \in B_n(r)} |c(x)(\frac{\partial A}{\partial z} \circ \alpha)(x)|$.*

Before the case $n = 1$ is established and the induction hypothesis is applied we need some preparations. As above put $\tilde{F} = (f_1, \ldots, f_m)$. Since $R_L \circ \tilde{F} \neq 0$ and $P_L(\tilde{F}, \tilde{f}) = 0$ it holds $\frac{\partial P_L}{\partial z}(\tilde{F}, \tilde{f}) \neq 0$. By the Weierstrass preparation theorem, after the generic linear change of the coordinates in $\mathbf{C}^n$ there are open neighborhoods of zeroes E, E' in $\mathbf{C}^{n-1}, \mathbf{C}$ respectively such that $\frac{\partial P_L}{\partial z}(\tilde{F}, \tilde{f})(x) = \hat{H}(x) W(x)$, $x = (x', x_n) \in E \times E' \subset \mathbf{C}^{n-1} \times \mathbf{C}$, for some non-vanishing function $\hat{H} \in \mathcal{O}(E \times E')$ and some unitary polynomial $W \in \mathcal{O}(E)[x_n]$ such that $W^{-1}(0) \cap (E \times \partial E') = \emptyset$.

Dividing $f_j, \tilde{f}$ by W to the power 2 one obtains:

$$f_j(x) = H_j(x) W(x)^2 + r_j(x),$$
$$\tilde{f}(x) = \tilde{H}(x) W(x)^2 + \tilde{r}(x)$$

for $x = (x', x_n) \in E \times E'$, where $r_j(x), \tilde{r}(x) \in \mathcal{O}(E)[x_n]$ satisfy $deg(r_j), deg(\tilde{r}) < deg(W^2)$ and $H_j, \tilde{H} \in \mathcal{O}(E \times E')$ for $j = 1, \ldots, m$.

Denote $d = deg(W)$. The polynomials $W, r_j, \tilde{r}$, for $j = 1, \ldots, m$, are of the form:

$W(x) = x_n^d + x_n^{d-1} a_1(x') + \ldots + a_d(x')$,
$r_j(x) = x_n^{2d-1} b_{j,0}(x') + x_n^{2d-2} b_{j,1}(x') + \ldots + b_{j,2d-1}(x')$,
$\tilde{r}(x) = x_n^{2d-1} c_0(x') + x_n^{2d-2} c_1(x') + \ldots + c_{2d-1}(x')$.

Replacing the holomorphic coefficients

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1}$$

for all j in $W, r_j, \tilde{r}$ by new variables denoted by the same letters we obtain polynomials $C, w_j, \tilde{w}$ respectively. Define:

$$\alpha_j = C^2 S_j + w_j, \beta = C^2 \tilde{S} + \tilde{w}$$

for $j = 1, \dots, m$ where $S_j, \tilde{S}$ are new variables. Now divide $P_L(\alpha_1, \dots, \alpha_m, \beta)$ by C^2 (treated as a polynomial in x_n with polynomial coefficients) and divide $\frac{\partial P_L}{\partial z}(\alpha_1, \dots, \alpha_m, \beta)$ by C to obtain

(*) $\quad P_L(\alpha_1, \dots, \alpha_m, \beta) = \tilde{W} C^2 + x_n^{2d-1} T_1 + x_n^{2d-2} T_2 + \dots + T_{2d},$

(**) $\quad \frac{\partial P_L}{\partial z}(\alpha_1, \dots, \alpha_m, \beta) = \bar{W} C + x_n^{d-1} T_{2d+1} + x_n^{d-2} T_{2d+2} + \dots + T_{3d}$, where $\tilde{W}, \bar{W}, T_1, \dots, T_{3d}$ are polynomials such that $T_1, \dots, T_{3d}$ depend only on the variables

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1}.$$

Let

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1},$$

where $j = 1, \dots, m$, again denote the holomorphic coefficients of $W, r_1, \dots, r_m, \tilde{r}$ as at the beginning. The tuple consisting of all these functions satisfies the system of equations $T_1 = \dots = T_{3d} = 0$ and may be uniformly approximated in some neighborhood of $0 \in \mathbf{C}^{n-1}$ by a sequence (indexed by ν) of tuples of Nash functions

$$a_1^\nu, \dots, a_d^\nu, b_{j,0}^\nu, \dots, b_{j,2d-1}^\nu, c_0^\nu, \dots, c_{2d-1}^\nu,$$

where $j = 1, \dots, m$, also satisfying the system $T_1 = \dots = T_{3d} = 0$. Indeed, if $n = 1$ then the coefficients are constant and therefore may be taken as their own approximations. If $n > 1$ we are done by the induction hypothesis because the coefficients depend on $n - 1$ variables.

Using the obtained Nash functions define:

$$W_\nu(x) = x_n^d + x_n^{d-1} a_1^\nu(x') + \dots + a_d^\nu(x'),$$
$$r_{j,\nu}(x) = x_n^{2d-1} b_{j,0}^\nu(x') + x_n^{2d-2} b_{j,1}^\nu(x') + \dots + b_{j,2d-1}^\nu(x'),$$
$$\tilde{r}_\nu(x) = x_n^{2d-1} c_0^\nu(x') + x_n^{2d-2} c_1^\nu(x') + \dots + c_{2d-1}^\nu(x').$$

Finally for, $j = 1, \dots, m$, define

$$f_j^\nu(x) = H_j^\nu(x)(W_\nu(x))^2 + r_{j,\nu}(x),$$

$$\bar{f}^\nu(x) = \tilde{H}^\nu(x)(W_\nu(x))^2 + \tilde{r}_\nu(x).$$

Here $H_j^\nu, \tilde{H}^\nu$, are any sequences of polynomials converging uniformly to $H_j, \tilde{H}$ in some neighborhood of $0 \in \mathbf{C}^n$ respectively.

Now it is easy to see that by (*), (**) and the way $f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu$ are defined, there is a neighborhood of $0 \in \mathbf{C}^n$ in which for every ν the following holds:

$$P_L(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = R^\nu(\frac{\partial P_L}{\partial z}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu))^2,$$

where $\{R^\nu\}$ is a sequence of holomorphic functions converging to zero. Therefore it suffices to apply Lemma 3.4 with $A = P_L(f_1^\nu, \ldots, f_m^\nu, z)$, $\alpha = \bar{f}^\nu$ and $c = R^\nu$ (for sufficiently large ν) to obtain

$$P_L(f_1^\nu, \ldots, f_m^\nu, \tilde{f}^\nu) = 0$$

in some neighborhood U_0 of zero for every $\nu \in \mathbf{N}$, where $\{\tilde{f}^\nu\}$ is a sequence of holomorphic functions converging to $\tilde{f}$ in U_0.■

3.2 Algorithm

Section 3.2 is devoted to a recursive algorithm of Nash approximation of a holomorphic mapping $F : U \to V \subset \mathbf{C}^{\hat{m}}$, where U is a neighborhood of zero in $\mathbf{C}^n$ and V is an algebraic variety. The correctness of the algorithm follows from the proof of Proposition 3.1.

The method of approximation presented here is more efficient than the one we developed in [8]. The main difference is that now the polynomial R_L defined in step 4 below need not be factorized into powers of pairwise distinct optimal polynomials.

For $\nu \in \mathbf{N}$, the approximating mapping $F^\nu = (f_1^\nu, \ldots, f_{\hat{m}}^\nu) : U_0 \to V$, returned as the output of the algorithm, is represented by $\hat{m}$ non-zero polynomials $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $i = 1, \ldots, \hat{m}$, such that $P_i^\nu(x, f_i^\nu(x)) = 0$ for $x \in U_0$. We restrict attention to the local case i.e. U_0 is an open neighborhood of a fixed $x_0 \in U$. More precisely, we work with the following data:

Input: a holomorphic mapping $F = (f_1, \ldots, f_{\hat{m}}) : U \to V \subset \mathbf{C}^{\hat{m}}$, $F = F(x)$, where U is an open neighborhood of $0 \in \mathbf{C}^n$ and V is an algebraic variety.

Output: $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $P_i^\nu \neq 0$ for $i = 1, \ldots, \hat{m}$ and $\nu \in \mathbf{N}$, with the following properties:

(a) $P_i^\nu(x, f_i^\nu(x)) = 0$ for every $x \in U_0$, where $F^\nu = (f_1^\nu, \ldots, f_{\hat{m}}^\nu) : U_0 \to V$ is a holomorphic mapping such that $\{F^\nu\}$ converges uniformly to F on an open neighborhood U_0 of $0 \in \mathbf{C}^n$,

(b) P_i^ν is a unitary polynomial in z_i of degree independent of ν whose coefficients (belonging to $\mathbf{C}[x]$) converge uniformly to holomorphic functions on U_0 as ν tends to infinity.

First let us comment on the notation and the idea of the algorithm. The meaning of the symbol $V_{(m)}$ and the notion of the optimal polynomial used below can be found in Subsection 2.2.

In steps 2 and 6 we apply linear changes of the coordinates. Having approximated the mapping $\hat{J} \circ F \circ J|_{J^{-1}(U)} : J^{-1}(U) \to \hat{J}(V)$, where $\hat{J} : \mathbf{C}^{\hat{m}} \to \mathbf{C}^{\hat{m}}$, $J : \mathbf{C}^n \to \mathbf{C}^n$ are linear isomorphisms, one can obtain the output data for F following standard arguments. (Composing F and J does not lead to any difficulties. As for $\hat{J}$, it is sufficient to use the fact that the integral closure of a commutative ring in another commutative ring is again a ring.) Therefore, when the coordinates are changed, we write what (as a result) may be assumed about the mapping F, but the notation is left unchanged.

The aim of steps 1-3 is to prepare the variety V so that for the polynomial P_L calculated in step 4, Lemma 3.2 holds (for details see the proof of Proposition 3.1). Steps 5-9 are responsible for the fact that $P_L(f_1^\nu,\ldots,f_m^\nu,\bar{f}^\nu) = R^\nu(\frac{\partial P_L}{\partial z}(f_1^\nu,\ldots,f_m^\nu,\bar{f}^\nu))^2$, where $f_1^\nu,\ldots,f_m^\nu$ are defined in step 10 whereas $\{R^\nu\}$ is a sequence of holomorphic functions converging to zero and $\{\bar{f}^\nu\}$ is a sequence of holomorphic functions converging to $L(f_{m+1},\ldots,f_{m+s})$. This enables to use Lemma 3.4 which together with Lemma 3.2 implies that $P_{m+1}^\nu,\ldots,P_{m+s}^\nu$ are well defined in step 11. As for $P_1^\nu,\ldots,P_m^\nu$, these polynomials are obtained in step 10 by applying the results of the algorithm switched for the lower dimensional case in step 9.

Algorithm: 1. Replace V by $V_{(m)}$ such that $F(U)\subset V_{(m)}$.
2. Apply a linear change of the coordinates in $\mathbf{C}^{\hat{m}}$ after which $\rho|_V$ is a proper mapping, where $\rho:\mathbf{C}^m\times\mathbf{C}^s\approx\mathbf{C}^{\hat{m}}\to\mathbf{C}^m$ is the natural projection.
3. Choose a $\mathbf{C}$-linear form $L:\mathbf{C}^s\to\mathbf{C}$ such that the generic fibers of $\rho|_V$ and $\tilde{\rho}|_{\Phi_L(V)}$ over $\mathbf{C}^m$ have the same cardinalities. Here $\tilde{\rho}:\mathbf{C}^m\times\mathbf{C}\to\mathbf{C}^m$ is the natural projection and $\Phi_L(y,v)=(y,L(v))$ for $(y,v)\in\mathbf{C}^m\times\mathbf{C}^s$.
4. Calculate the optimal polynomial $P_L(y,z)\in(\mathbf{C}[y])[z]$ describing $\Phi_L(V)\subset\mathbf{C}_y^m\times\mathbf{C}_z$. Calculate the discriminant $R_L\in\mathbf{C}[y]$ of P_L.
5. If $R_L(f_1,\ldots,f_m)=0$ then return to step 2 with m, s, V replaced by $m-1$, $s+1$, $V\cap\{R_L=0\}$ respectively. Otherwise put $\tilde{f}=L(f_{m+1},\ldots,f_{m+s})$ and observe that $\frac{\partial P_L}{\partial z}(f_1,\ldots,f_m,\tilde{f})\neq 0$.
6. Apply a linear change of the coordinates in $\mathbf{C}^n$ after which the following holds: $\frac{\partial P_L}{\partial z}(f_1,\ldots,f_m,\tilde{f})(x)=\hat{H}(x)W(x)$ in some neighborhood of $0\in\mathbf{C}^n$, where $\hat{H}$ is a holomorphic function, $\hat{H}(0)\neq 0$ and W is a unitary polynomial in x_n with holomorphic coefficients depending on $x'=(x_1,\ldots,x_{n-1})$. Put $d=deg(W)$.
7. Divide $f_i(x),\tilde{f}(x)$ by $(W(x))^2$ to obtain $f_i(x)=(W(x))^2H_i(x)+r_i(x)$ and $\tilde{f}(x)=(W(x))^2\tilde{H}(x)+\tilde{r}(x)$ in some neighborhood of $0\in\mathbf{C}^n$, $i=1,\ldots,m$. Here $H_i,\tilde{H}$ are holomorphic functions and $r_i,\tilde{r}$ are polynomials in x_n with holomorphic coefficients depending on x', such that $deg(r_i),deg(\tilde{r})<2d$.
8. Treating $H_i,\tilde{H}$, $i=1,\ldots,m$, and all the coefficients of $W,r_1,\ldots,r_m,\tilde{r}$ as new variables (except for the coefficient 1 standing at the leading term of W) apply the division procedure for polynomials to obtain:
$P_L(W^2H_1+r_1,\ldots,W^2H_m+r_m,W^2\tilde{H}+\tilde{r})=$
$=\tilde{W}W^2+x_n^{2d-1}T_1+x_n^{2d-2}T_2+\ldots+T_{2d}$,
$\frac{\partial P_L}{\partial z}(W^2H_1+r_1,\ldots,W^2H_m+r_m,W^2\tilde{H}+\tilde{r})=$
$=\bar{W}W+x_n^{d-1}T_{2d+1}+x_n^{d-2}T_{2d+2}+\ldots+T_{3d}$.
Here $T_1,\ldots,T_{3d}$ are polynomials depending only on the variables standing for the coefficients of W, $r_1,\ldots,r_m,\tilde{r}$. Moreover, $T_1\circ g=\ldots=T_{3d}\circ g=0$, where g is the mapping whose components are all these coefficients (cp. Section 3.1).
9. If $n>1$ then apply the Algorithm with F,V replaced by g,$\{T_1=\ldots=T_{3d}=0\}$ respectively. As a result, for every component $c(x')$ of $g(x')$ and for every $\nu\in\mathbf{N}$ one obtains a unitary polynomial $Q_c^\nu(x',t_c)\in(\mathbf{C}[x'])[t_c]$ which put in place of $P_i^\nu(x,z_i)$ satisfies (a) and (b) above with x, z_i, f_i^ν replaced by x', t_c, c^ν respectively. Here, for every c, $\{c^\nu\}$ is a sequence of Nash functions converging

to c, in some neighborhood of $0 \in \mathbf{C}^{n-1}$, such that for every fixed ν the mapping g^ν obtained by replacing every c of g by c^ν satisfies $T_1 \circ g^\nu = \ldots = T_{3d} \circ g^\nu = 0$. If $n = 1$ then g is constant and then it is its own approximation yielding the Q^ν_c's immediately.

10. Approximate H_i for $i = 1, \ldots, m$, by a sequence $\{H^\nu_i\}$, of polynomials. Let $W_\nu, r_{1,\nu}, \ldots, r_{m,\nu}, \tilde{r}_\nu$, for every $\nu \in \mathbf{N}$, be the polynomials in x_n defined by replacing the coefficients of W, $r_1, \ldots, r_m, \tilde{r}$ by their Nash approximations (i.e. the components of g^ν) determined in step 9. Using Q^ν_c (for all c) and H^ν_i one can calculate $P^\nu_i \in (\mathbf{C}[x])[z_i]$, for $i = 1, \ldots, m$, satisfying (b) and (a) with $f^\nu_i = H^\nu_i(W_\nu)^2 + r_{i,\nu}$ being the i'th component of the mapping F^ν (whose last $\hat{m} - m$ components are determined by $P^\nu_{m+1}, \ldots, P^\nu_{\hat{m}}$ obtained in the next step). To calculate $P^\nu_1, \ldots, P^\nu_m$ one can follow the standard proof of the fact that the integral closure of a commutative ring in another commutative ring is again a ring.

11. Put $V^\nu = \{(x, z) \in \mathbf{C}^n_x \times \mathbf{C}^{m+s}_z : z \in V, P^\nu_i(x, z_i) = 0 \text{ for } i = 1, \ldots, m\}$, where $z = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+s})$. For $i = 1, \ldots, s$ and $\nu \in \mathbf{N}$ take $P^\nu_{m+i} \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν onto $\mathbf{C}^n_x \times \mathbf{C}_{z_{m+i}}$.

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